Engineering Notes

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Nature of Coupling in Nonconservative Lumped Parameter Systems

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Introduction

ALTHOUGH the response of undamped or at least proportionally damped linear systems can be determined easily by modal analysis, the prediction of the response of nonproportionally damped linear systems requires either more extensive computations or some convenient assumptions and approximations. This happens as a result of the damping matrix and stiffness matrix having different eigenvectors. The modes of the damped structure then become complex for underdamped systems, and the equations cannot be decoupled.

The solution of the time response of most multiple degree-of-freedom systems can be calculated fairly efficiently by casting the problem in state space form and using a sophisticated Runge-Kutta method. However, it is often an advantage in design work to examine the vibration problem in the second-order form retaining the physical description of position, velocity, and acceleration. Modal coordinates are particularly intuitive for design work and provide for straightforward comparisons with experimentally measured quantities. In this sense, the modal approach also provides for the possibility of including nonlinear effects and frequency-dependent damping. State space methods are of twice the dimension and lose the advantage of symmetry and definiteness of the structure described in physical coordinates. In addition, many systems have banded coefficient matrices, something which is lost in the first-order state-space approach.

For all of the reasons mentioned, various methodologies have been proposed in the past dealing with nonnormal mode systems.[1-4] All of them are accurate enough in specific applications, but they are limited by several factors such as the level of damping and the separation of modes. Therefore, each of these approximate methods is insufficient in analyzing and explaining the general underdamped linear finite-dimensional vibration problem. Recent efforts have been centered around introducing indices to measure the nonproportionality of a given system and not in deriving an approximate response.[5,6]

The primary objective of this Note is to present an approximate method that, by performing simple calculations, takes into account the coupling between the modes and leads to a closed-form solution concerning the response of the system. The proposed decoupling process, subject to practically no restrictions, is simple enough to conceive and accurate and reliable enough to apply. The development of the nonproportionality indices presented here focuses on the measurement of the modal coupling existing in the system and the qualitative prediction of the system behavior.

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System Equations and Modal Analysis

The class of problems considered here are those machines and structures that can be successfully modeled by the vector differential equation

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = F(t), \qquad x(0) = x_0, \quad \dot{x}(0) = v_0 \quad (1)$$

Here M, D, and K are symmetric and positive definite $n \times n$ real matrices representing the lumped mass, viscous damping, and stiffness characteristics of the system, $x(t)$ and $F(t)$, respectively, denote the $n \times 1$ response and excitation vector, x_0 and v_0 $n \times 1$ vectors form the displacement and velocity initial conditions, and the overdots represent differentiation with respect to time.

Consider the transformation to modal, or natural, coordinates:

$$x(t) = M^{-1/2}\Phi n(t) = \Psi n(t) \quad (2)$$

where $M^{-1/2}$ is the inverse matrix square root of the mass matrix, and $n(t)$ is the $n \times 1$ modal, or natural, coordinate vector. The $n \times n$ matrices Φ and Ψ represent the modal matrix of the corresponding undamped system, normalized with respect to the identity and mass matrix, respectively. Substituting Eq. (2) into Eq. (1) and premultiplying Eq. (1) by Ψ^T, the system equations, expressed in the natural coordinate space $n(t)$, are given by

$$I\ddot{n}(t) + \tilde{D}\dot{n}(t) + \tilde{K}n(t) = \tilde{F}(t) \quad (3)$$

Here, I is the identity matrix, $\tilde{D} = \Psi^T D\Psi$, $\tilde{K} = \Psi^T K\Psi$, $\tilde{F}(t) = \Psi^T F(t)$, and Ψ^T denotes the transpose of the matrix Ψ.

Obviously, the matrix $\tilde{K}$ is diagonal, with diagonal elements equal to the squared undamped natural frequencies. If the matrix $\tilde{D}$ was also diagonal, the system under study would possess normal modes. Unfortunately, this is not always the case. In fact, the matrices D and K belong to the same eigenvector space and, therefore, the system is diagonalizable, if and only if the commutivity condition

$$DM^{-1}K = KM^{-1}D \qquad \text{or} \qquad \tilde{D}\tilde{K} = \tilde{K}\tilde{D} \quad (4)$$

holds.[7]

In general, Eq. (3) represents a system of coupled linear differential equations and can be expressed as

$$\ddot{n}_i(t) + \sum_{j=1}^{n} \tilde{d}_{ij}\dot{n}_j(t) + \omega_i^2 n_i(t) = \tilde{F}_i(t), \qquad i = 1,2,\dots,n \quad (5)$$

Decoupling Process

An approximate decoupling process is proposed here, leading to a closed-form solution for the response of the system characterized by coupled modes.

Equation (5) also can be written as

$$\ddot{n}_i(t) + \{\tilde{d}_{ii} + T_i\}\dot{n}_i(t) + \omega_i^2 n_i(t) = \tilde{F}_i(t) \qquad i = 1,2,\dots,n \quad (6)$$

where the extra terms

$$T_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \tilde{d}_{ij} \left[\frac{\dot{n}_j(t)}{\dot{n}_i(t)} \right], \quad i = 1,2,\ldots,n \tag{7}$$

intrinsically impose the modal coupling and indicate how, quantitatively and qualitatively, one mode affects the other during the vibration process.

Let the index s denote the steady-state response properties caused by harmonic excitation with frequency $\omega = \omega_s$. In the complex domain, the excitation can be expressed as

$$\tilde{F}_{is}(t) = f_{is} e^{J\omega_s t}, \qquad i = 1,2,\ldots,n \tag{8}$$

where $J = (-1)^{1/2}$ represents the imaginary unity, and $e = 2.7183$ is the base of the natural logarithms.

If Eq. (5) were completely decoupled, it could be written in the familiar modal form

$$\ddot{n}_{is}(t) + 2\xi_{is}\omega_i \dot{n}_{is}(t) + \omega_i^2 n_{is}(t) = \tilde{F}_{is}(t), \qquad i = 1,2,\ldots,n \tag{9}$$

Here ξ_{is}, $i = 1,2,\ldots,n$ are defined to be fictitious damping ratios calculated by either using experimentally obtained data, as for example the transfer function properties,[1] or by using analytical expressions.[3,9] For a proportionally damped system, the analytical expressions should enable the fictitious damping ratios to render the exact modal damping ratios.

According to the Frequency Response Theorem,[8] the steady-state natural coordinate response will have the form

$$n_{is}(t) = n_{is} e^{(\omega_s t - \phi_{is})J}, \qquad i = 1,2,\ldots,n \tag{10}$$

Substituting Eq. (10) into Eq. (9) and after some simple manipulations, the amplitude and phase decoupling parameters N_{is}, ϕ_{is}, respectively, are found to be

$$N_{is} = \frac{f_{is}}{\sqrt{(\omega_i^2 - \omega_s^2)^2 + (2\omega_i\omega_s\xi_{is})^2}} \tag{11a}$$

$$\phi_{is} = \tan^{-1}\left[\frac{2\omega_i\omega_s\xi_{is}}{\omega_i^2 - \omega_s^2}\right], \qquad i = 1,2,\ldots,n \tag{11b}$$

Now, assuming that the coupling ratios can be approximated by their steady-state coupling formulation, then the steady-state-based coupling terms can be expressed as

$$T_{is} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \tilde{d}_{ij} \left[\frac{N_{js}}{N_{is}}\right] e^{-(\phi_{js} - \phi_{is})J}, \qquad i = 1,2,\ldots,n \tag{12}$$

and Eq. (6) can be written as

$$\ddot{n}_i(t) + \{\tilde{d}_{ii} + T_{is}\}\dot{n}_i(t) + \omega_i^2 n_i(t) = \tilde{F}_i(t)$$
$$i = 1,2,\ldots,n \tag{13}$$

where the terms T_{is}, $i = 1,2,\ldots,n$ are given by Eq. (12). If the commutivity condition of Eq. (4) were satisfied, then the terms T_{is} would be zero. In the case where Eq. (4) is not satisfied, the terms T_{is} represent a correction to the decoupled modal equations. Hence, Eq. (13) provides an approximation of the exact solution involving only the calculation of the undamped eigenvalues and eigenvectors.

Note that the entire development was based on a harmonic type of excitation, i.e., $\tilde{F}_i(t) = \tilde{F}_{is}(t)$, $i = 1,2,\ldots,n$. However, the results can be generalized, and it can be shown that Eq. (13) holds for any piecewise continuous function of time as excitation.

Nonproportionality Indices

It becomes obvious from Eq. (13) that the ratios

$$c_{is} = \frac{T_{is}}{\tilde{d}_{ii} + T_{is}}, \qquad i = 1,2,\ldots,n \tag{14}$$

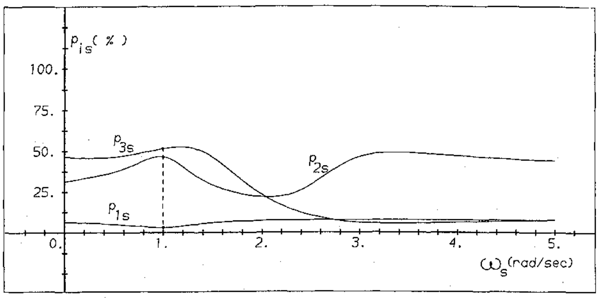

Fig. 1 Frequency spectrum of nonproportionality indices.

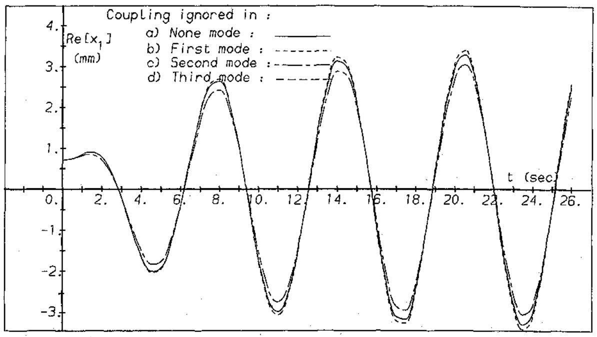

Fig. 2 Real part of the response at DOF-1 vs time. Excitation with frequency: ω_s = 1.0 rad/s.

could provide both a qualitative and quantitative sense of the amount of coupling involved in each mode for excitation frequency ω_s. However, their complex nature requires that the nonproportionality indices be defined as

$$p_{is} = |c_{is}| = \left| \frac{T_{is}}{\tilde{d}_{ii} + T_{is}} \right|, \qquad i = 1,2,\ldots,n \tag{15}$$

where the coupling terms T_{is}, $i = 1,2,\ldots,n$ are given by Eqs. (12). Notice that the above indices are implicitly dependent on the excitation frequency ω_s. Consequently, as it is shown in the example, they can be used to visualize the modal coupling in a frequency spectrum.

A "good proportional damping approximation" criterion is

$$\frac{1}{\omega_{\max}} \int_0^{\omega_{\max}} p_{is}(\omega_s)\, d\omega_s \ll 1 \tag{16}$$

When this criterion is satisfied, there is only a small amount of coupling involved in the corresponding mode. Therefore, the respective off-diagonal terms of the transformed damping matrix $\tilde{D}$ can be neglected, and the respective equation (13) can be approximated by an uncoupled equation.

It is emphasized that the basic difference of the indices developed in this work from a similar index given by other authors[5] is that the summations are not only over the off-diagonal terms, but over a combination of off-diagonal terms and modal coupling terms [see Eq. (12)].

Example and Conclusions

Consider a three-degree-of-freedom linear vibratory system under harmonic type of excitation, modeled as

$$\begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} 0.540 & -0.240 & -0.141 \\ -0.240 & 0.740 & -0.141 \\ -0.141 & 0.141 & 1.200 \end{bmatrix} \dot{x}(t)$$

$$+ \begin{bmatrix} 1.105 & -0.105 & 0.000 \\ -0.105 & 1.105 & 0.000 \\ 0.000 & 0.000 & 9.000 \end{bmatrix} x(t) = \begin{bmatrix} 1.0 \\ 2.0 \\ 1.0 \end{bmatrix} e^{J\omega_s t}$$

$$x(0) = [0.707\ 0.707\ 0.000]^T, \quad \dot{x}(0) = [0.000\ 0.000\ 0.000]^T$$

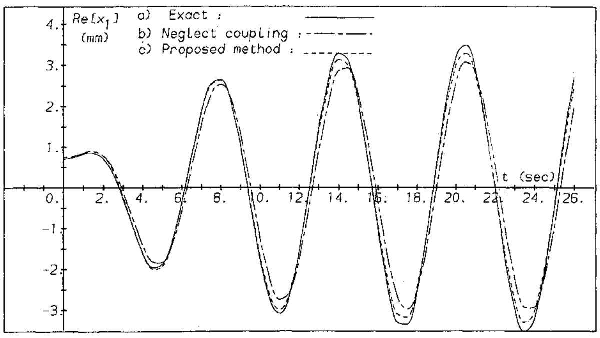

Fig. 3 Real part of the response at DOF-1 vs time. Excitation with frequency: $\omega_s = 1.0$ rad/s.

The eigenproblem of the undamped system yields natural frequencies $\omega_1 = 1.0$, $\omega_2 = 1.1$, and $\omega_3 = 3.0$ (ω_1 and ω_2 closely spaced), and modal matrix

$$\Psi = \begin{bmatrix} 0.707 & -0.707 & 0.000 \\ 0.707 & 0.707 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}$$

Note that $DM^{-1}K \neq KM^{-1}D$ and, therefore, the system under study does not possess normal modes, i.e., it is not diagonalizable. In order to form the fictitious damping ratios, we applied the formula

$$\xi_{is} = \frac{\sum_{j=1}^{3} \tilde{d}_{ij}}{2(\omega_i \omega_s)^{1/2}}, \qquad i = 1,2,3$$

which leads to the classical definition of the damping ratios for a proportionally damped system, i.e., for $\tilde{d}_{ij} = 0$ $(i \neq j)$ and $\omega_s = \omega_i$. Notice here that the proposed method is capable of handing problems with frequency-dependent damping ratios as, for example, in several applications of viscoelastic materials with frequency-dependent damping properties.

In Fig. 1 we present the frequency spectrum of nonproportionality indices. By inspection one can easily realize where the modal coupling becomes an important factor, and when the off-diagonal terms of the transformed damping matrix $\tilde{D}$ are negligible.

From the coordinate transformation [Eq. (2)] and the form of the previously given modal matrix Ψ, we conclude that only the first two modes participate in each of the first two degrees of freedom, and only the third mode in the third degree of freedom. To be more concise and maintain the physical sense, we illustrate next only the real part of the response. In Fig. 2 we present the response of the first degree of freedom for excitation frequency $\omega_s = \omega_1 = 1.0$ rad/s. As the nonproportionality indices predict (see Fig. 1), the highest deviations occur when one ignores the coupling in the dominant second mode. In fact, it is clear that we can neglect the coupling in the first mode throughout the frequency range without introducing significant errors.

Figure 3 was formed to compare the solution given by our method, to the exact solution as well as to the one obtained by neglecting the coupling completely. Note that our proposed approximate method results in steady-state responses close to the exact ones, and corrects those responses resulting by simply neglecting the off-diagonal terms of the transformed damping matrix $\tilde{D}$, where the consideration of the modal coupling becomes important. Deviations from the exact solution should occur in the transient part of the response, due to the decoupling procedure we used. The primary advantage of the proposed method is the computational simplicity (recalling that the state formulation of the problem requires matrices and vectors belonging to a $2n$-dimensional space).

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Bounded-Input/Bounded-Output Stability of Linear Multidimensional Time-Varying Systems

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Introduction

In recent years, the attention of several authors has been attracted by the stability of systems governed by the equation

$$M(t)q''(t) + G(t)q'(t) + K(t)q(t) = B(t)v(t) \qquad (1)$$

where primes denote differentiation with respect to time t, the independent variable; q is an $(n \times 1)$ vector; M, G, K are $(n \times n)$ real matrices (whose elements are functions of t and differentiable); B is an $(n \times m)$ real matrix (whose elements are functions of t); v is an $(m \times 1)$ vector; and n is the dimension of the system (which is often large).

Such systems are encountered in spacecraft dynamics, economics, ecology, biosystems, demography, and several engineering disciplines. With the exception of trivial cases, an explicit solution of Eq. (1) is impossible. The only way to

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